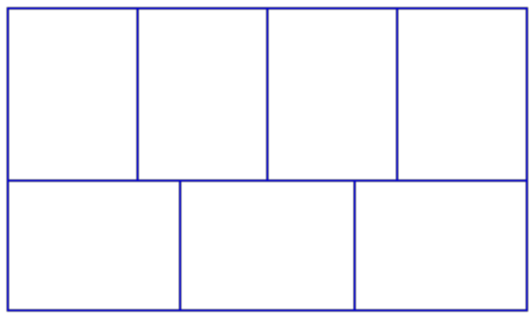
NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Problem of the Week Teacher Packet

Congruent Rectangles

These seven congruent rectangles form a larger rectangle. If the area of the larger rectangle is 756 units^2, what is its perimeter?



Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually get the answer) we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

The perimeter of the rectangle is 114 units.

If your answer does not match our answer,

- don't forget to use the small rectangles in your solution – they do matter!
- can you figure out the area of each of the small rectangles?
- notice that the horizontal line across the large rectangle is made of four widths of the small rectangle *or* three lengths of the small rectangle – four widths must equal three lengths.
- you can assign variables to dimensions of the small rectangle and use algebra to solve the problem, but algebra isn't required.

If these ideas helps you, you might revise your answer, and then leave a comment that tells us what you did. If you're still stuck, leave a comment that tells us where you think you need help.

If your answer does match ours,

- have you said *why* you can do each step you did?
- did you make any mistakes along the way? If so, how did you find them?
- what hints would you give another student?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did—you might answer one or more of the questions above.

Our Solutions

Method 1: Guess and Check, Area of Small Rectangles

Since all the small rectangles are congruent, they all have the same area. 756/7 = 108 units^2. This means that the dimensions of the small rectangles need to multiply to 108. I made a chart of possible factor pairs (I'm assuming the dimensions are integers, and will see if it works).

1, 108
2, 54
3, 36
4, 27
6, 18
9, 12

I took the first pair, 1 and 108, and wrote 108 next to each long side of the small rectangle in the picture across the bottom, and 1 next to each short side of the small rectangle across the top. Since it's a rectangle, those two sides have to be equal. It was obvious that the four 1s across the top weren't nearly long enough to match the three 108s across the bottom.

I decided to try 6 and 18. This made the top 24 and the bottom 48. Not equal.

I tried 9 and 12. That makes the top 36 and the bottom 36. They're equal! So the dimensions of the small rectangles are 9 and 12. If I label the rest of the large rectangle and find the distance all the way around, I find that the perimeter is 114 units.

(*Note:* Students might more directly see that the dimensions of the small rectangle have to be in a ratio of 3:4, because the top and bottom of the large rectangle must be equal. Then they might test the factor pairs of 108 until they find one that has a ratio of 3:4 – it's a more abstract way of doing just what was done in this example.)

Method 2: Guess and Check, Ratio of the Dimensions of the Small Rectangle

I noticed that there are four short sides of the small rectangle across the top of the large rectangle and three long sides of the small rectangle across the bottom. This means that four of the short ones must equal three of the long ones, since the opposite sides of a rectangle are equal.

I also noticed that the area of the small rectangle is 108 units^2, since there are seven small rectangles and they are all congruent (meaning they are equal). 756 divided by 7 is 108.

I decided to pick pairs of numbers with a ratio of 3:4 and see if they multiply to 108, since the short and long sides of the small rectangles must multiply to 108, since length times width gives the area of a rectangle. I made a table of the numbers and their products:

3:4	12
6:8	48
9:12	108

The last choice worked, so 9 and 12 must be the dimensions of the small rectangles. Adding these up all the way around the large rectangle gives a perimeter of 114 units.

Method 3: System of Two Equations with Two Unknowns

Assign x as the width of the small rectangle and y as the length. The top and bottom of the big rectangle are equal, and the top is 4x while the bottom is 3y. That gives us

$$4x = 3y$$

Also, we know that the area is 756 u^2, and the dimensions of the large rectangle are 4x and x + y (since the edge is a length and a width of the small rectangle). That gives us

$$4x(x + y) = 756$$

In the first equation, we solve for y to get

$$\frac{4x}{3} = y$$

Substituting this into the second equation, we solve for x.

$$4x(x + \frac{4x}{3}) = 756$$

$$4x(\frac{3x + 4x}{3}) = 756$$

$$4x(\frac{7x}{3}) = 756$$

$$28x^2 = 2268$$

$$x^2 = 81$$

$$x = 9, -9$$

We know that x can't be -9 because we are talking about lengths of segments, and lengths cannot be negative. So x = 9. We go back to the third equation and substitute 9 in for x:

$$\frac{4 * 9}{3} = y$$

$$12 = y$$

The dimensions of the small rectangle are 9 and 12. Now we find the perimeter of the large rectangle, the formula for which is 2(L + W). Remember that the top of the large rectangle is 4x and length of the sides is x + y.

$$\begin{aligned} perimeter &= 2(4x + (x + y)) \\ &= 2(36 + 21) \\ &= 2(57) \\ perimeter &= 114 \end{aligned}$$

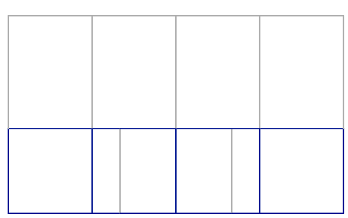
So the perimeter of the large rectangle is 114 units.

(*Note:* Students might do the same thing using the area of the small rectangle instead of the large one, which leads to slightly simpler algebra.)

Method 4: Direct Calculations of Dimensions of Small Rectangle

I noticed that each small rectangle must have an area of 108 units^2, since there are seven congruent rectangles, and 756/7 = 108. This means that the three rectangles across the bottom of the large rectangle have a total area of 324 units^2.

I extended the lines forming the four small rectangles at the top to split the bottom portion of the large rectangle into four squares:



The gray lines are the original dividers. These four squares have a total area of 324 units^2, so each has an area of 81 units^2. Since they're squares, that means they must have an edge length of 9 units. So the short side of the small rectangles is 9 units long. Since the area of a small rectangle is 108 units^2, the other side must be 12 units long. This results in a perimeter of 114 units for the large rectangle.

Standards

If your state has adopted the Common Core State Standards, you might find the following alignments helpful. Being familiar with and using these relationships comes up in middle school, whereas in High School Geometry, students are expected to be able to prove them.

Grade 4: Measurement and Data

3. Apply the area and perimeter formulas for rectangles in real world and mathematical problems.

Grade 8: Geometry

Understand congruence and similarity using physical models, transparencies, or geometry software.

Grade 8: Expressions and Equations (optional)

8c. Solve real-world and mathematical problems leading to two linear equations in two variables.

High School: Geometry: Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
7. Look for and make use of structure.

Teaching Suggestions

This is a problem that we have used in workshops with teachers and with many groups of students. It's a great problem with which to use our "Scenario Only" approach, either by not telling students that they are looking for the perimeter, or by not even giving them the area of the large rectangle at first until they have found as many relationships in the drawing as possible. (Versions of these two possible scenarios are available in the resources available with the problem when you are logged in as a teacher.)

One common mistake students will make is to ignore the small rectangles. This leads to a number of possible perimeters for the large rectangle, 110 units being the most popular (from dimensions of 27 and 28). I've suggested to these students that they write their longer dimension on the top and bottom of the rectangle, then figure out the resulting dimensions of the small rectangle. For example, for the student who uses 28 as the longer dimension, this implies that the short side of the small rectangle is 28/4, or 7, while the longer dimension is 28/3. Now add those two dimensions to find the short dimension of the large rectangle – in this case, 7 + 28/3, which is about 49/3, or around 16. This isn't anywhere near the 27 they clamed it was. Presenting students with an opportunity to find inconsistencies in their own work is often motivating. It also gives them some ideas about how they might check their own work to see if it makes sense.

Another thing that happens is that students find the area of the small rectangles to be 108 units^2, but then just say the dimensions must be 9 units and 12 units, without explaining why. I often ask these students if we could have also picked 6 and 18 as the dimensions for the small rectangle – it's often easier to explain why something *doesn't* work than why something *does*.

Other students might not know how to find the area or perimeter of a rectangle, not understand what "congruent" means, or simply not know how to dive in – that's the impetus for the "Scenario Only" plan of attack, combined with some "noticing and wondering" to get things moving (see the first two rounds of the Activity Series for more information about this idea).

Sample Student Solutions – Focus on **Interpretation, Strategy, and Completeness**

In the solutions below, I've focused on three different categories in the rubric – either Interpretation, Strategy, or Completeness – depending on which area I think the student most needs to concentrate. As a result, all of the submissions are scored either Novice or Apprentice in one of those categories, and my comments are designed to help them improve in that area.

	Novice	Apprentice	Practitioner	Expert
Interpretation	Shows understanding of one or none of the criteria in the Practitioner column.	Shows understanding of two of the criteria in the Practitioner column. Doesn't seem to understand what it means to solve in terms of x. Might use a specific value for x.	Understands: • all of the angle relationships provided and implied in the problem • that the goal is to find the measure of angle BCD • that the answer isn't a single number, but that it will be in terms of x	Is at least a Practitioner in Strategy and has answered the Extra question correctly.
Strategy	Has no ideas that will lead them toward a successful solution. Seems to rely solely on intuition.	Uses a strategy that uses luck instead of skill, or doesn't provide enough detail to determine.	Uses a strategy that relies on skill, not luck, which might include • thoroughly labeling a picture • adding an auxiliary line (the altitude of the triangle) • using angle relationships in triangles and possibly with parallel lines/transversals	Solves the problem two different ways.
Accuracy	Has made many errors.	Has made several mistakes or misstatements.	Makes few mistakes of consequence and uses largely correct vocabulary. Uses correct units when they use units.	[Generally not possible – can't be more accurate than Practitioner.]
Completeness	Has written very little that explains how the answer was achieved.	Submitted explanation without work or work without explanation. Leaves out enough details that a fellow student couldn't follow or learn from the explanation.	Explains most of the steps taken to solve the problem, and the rationale for them, with enough detail for another student to understand. This might include • explaining how they know they have the correct side lengths for the small rectangles	Adds in useful extensions and further explanation of some of the ideas involved. The additions are helpful, not just "I'll say more to get more credit".

Christine, age 15, Novice in Interpretation

The perimeter of the larger rectangle is 126 units.

Since there are seven smaller rectangles, I divided 756 by 7, and I got 18. Then, to find the total of all the sides, I multiplied 18 by 7 to get 126.

It's not clear what Christine actually does know. It seems like she realizes that the small rectangles all have the same area (which is 108). But she writes 18 instead of 108, then multiplies that by 7. It makes me feel like she has organizational problems as much as conceptual problems. I might start by asking her to check her answer for 756/7, then tell me what the result actually represents.

Kacy, age 15, Novice in Interpretation

The perimeter of the larger rectangle is 189 units.

I divided the area by the number of sides of the rectangle which is four to get the perimeter.

Kacy seems confused about the difference between area, perimeter, and side length. I might ask her to tell me the area and perimeter of a rectangle with sides of 4 and 6 just to learn more about what she knows about those two ideas.

Jason, age 16, Novice in Strategy

The perimeter is of this rectangle is 110 units.

Using the idea that there are 7 equal rectangles I tried to figure out the measure of the top length of the rectangle. It was made up of four smaller rectangles so I wrote down the multiples of 7 and 4 and found out that 28 was the number that they had in common. After figuring out that 28 was the top figure I calculated 27 was the side figure. 28x27= 756. Perimeter=2L+2W. P=56+54=110 units.

I would love to know more about why he used multiples of 7 and 4! In fact, I might just ask him to say more about why the length of the top of the large rectangle must be a multiple of both 7 and 4.

Andy, age 14, Apprentice in Interpretation

The perimeter of the triangle is 150units.

I wasn't sure on a correct way to figure out the problem correctly, so I fell back and used the trusty method of guess and check. This metod worked and I found that 12 is one side and 63 is the other. I'm not sure if this is correct because I also found some other sides that work (27 and 28).

Andy hasn't said what he guessed or how he checked it, though it seems likely that he simply looked for numbers that multiply to 756. He's primed to think about this some more, since he doubts his work. I would ask him to figure out dimensions of the small rectangles, based on his 12 and 63, and see if it all works out (see the Teaching Suggestions section), and ask him to check 27 and 28 in the same way.

Nick, age 15, Apprentice in Interpretation

The Perimeter of this problem is 42units.

The way i did this problem is i took the area, 756units^2, and divided it by 7, since all the squares are congruent. when i divided it, i got the area for each square was 108. the next thing i did, was look at what two numbers equal to give you 108, n when you multiply the lower number by 4, and the greater number by 3 you get the same answer. I than found 12, and 9. I checked to see if 9x4=12x3, and i was correct. i than added 12+12+9+9, to get 42 units.

It seems like Nick has done a pretty good job of answering the wrong question. I would suggest that he reread the actual question to be sure he's answering the question. I would also ask him to say more about why he's multiplying one number by 4 and the other by 3, since that's important information.

Andrea, age 16, Apprentice in Strategy

I'm really stuck, please help...

I've been working on this for quite some time. I started out trying to make an equation that was (L+W)(3W)=756 which comes out to be in the next step 3L+LW+3W+W(to the 2nd power) =756 but i dont know what to do with that Am I going in a completely wrong direction with this? Then i tried dividing 7 from the area of 756 to get the area of each congruent rectangle, and I got 108, but I dont know what to do with that. Please help me

Andrea has made some arithmetic errors when distributing through her parentheses. But more importantly, she doesn't realize that she needs two equations that relate L and W. I would point her in that direction, and only then would I help her work on her variable arithmetic.

Arash, age 16, Apprentice in Strategy

Perimeter=1197

length of each rectangle= "X" width of each rectangle= "Y"
(Y+Y+Y+Y)(X+Y)=756 => (4Y)(X+Y)=756

(X+X+X)(X+Y)=756 => (3X)(X+Y)=756

==> (4Y)(X+Y)=(3X)(X+Y)=756

==> 4Y=3X=756

==> Y=189 ; X=252

==> PERIMETER=Y+Y+Y+Y+X+Y=5Y+X=(5)(189)+252=1197

Arash has written two true equations for the area of the large rectangle, but then comes up with an incorrect statement (writing 4Y=3x=756). I would point that out to him, as he's lost the (X+Y) term from the first two statements. I might also suggest that he could check his answer by finding the length and the width of the large rectangle, using 189 and 252, and see if that results in an area of 756 units^2.

Jake, age 15, Apprentice in Strategy

The perimiter is 114 units.

I divided the 756 units squared by 7 to get the area of each smaller rectangle. Then I came up with 108. I knew that 12 and 9 multiplied together to make 108 so I concluded that was the length of the sides

Jake hasn't said anything about how he knows 9 and 12 are the appropriate factor pair of 108 to choose. I would ask him how he knows that's the right one, and whether we could also use 6 and 18.

Mary, age 15, Apprentice in Completeness

The total perm. is 114, where the legnth is 36 and the width is 21.

If you guess and check and find 36x21 then you see that 21x36=756.
So 21+21+36+36=114, the found perm.

Mary definitely needs to say more about how she decided on 36 and 21 – as it is, we can't be sure she didn't get lucky. I would ask her why she picked that pair, and whether she could also use two numbers like 27 and 28.

Sean, age 14, Apprentice in Completeness

The perimeter of the larger rectangle is 114 units.

x.y=756
765/7=108
a.b=108
x=3a
x=4b
b=3/4a
y=a+b
3/4a^2=108
a^2=114
a=12
b=9
x=36
y=21
perimeter=36+36+21+21

" "=114

This is a classic case of showing your work but not explaining any of it. I would ask him to start by explaining what his variables represent, and where his first five lines come from.

Joe, age 16, Apprentice in Accuracy

The perimeter is 366 units.

I divided the area by 7 to find the area for each smaller rectangle. Then i used the formula 4x(x+108/x)=756 to find x. I then substituted the value of x into the formula for the perimeter 2(4x)+2(x+108/x) and found that the perimeter is 366.

It looks like Joe has chosen a good strategy, but it seems like he has used 36 for x (instead of the 9 he should have gotten from his first equation). I would ask him to show me the work he did in solving the first equation and to show the value he got for x.

Scoring Rubric

A **problem-specific rubric** can be found linked from the problem to help in assessing student solutions. We consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work.

We hope these packets are useful in helping you make the most of Geometry Problems of the Week. Please let me know if you have ideas for making them more useful.

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