NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

**Problem of the Week Teacher Packet**

# To B or Not To B

The first four terms of a pattern are shown below:

**ABBA**

**AABBBAA**

**AAABBBBAAA**

**AAAABBBBBAAAA**

Suppose that this pattern continues, with the As and Bs increasing as shown.

1. Find an expression for the number of Bs in the nth term of the pattern.
2. Find an expression for the ratio of Bs to total letters in the nth term.
3. Use your expression to determine which term has exactly 35% Bs.

**Extra:** Can you find a term in the pattern that has exactly one-third Bs and two-thirds As? Why or why not?

## Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually get the answer) we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

1. *There are n + 1 Bs in the nth term.*
2. *One possible answer for the ratio of Bs to total letters in the nth term is* $\frac{n+1}{n+3}$.
3. *The 13th term has exactly 35% Bs.*

If your answer does not match our answer,

- did you try making a table showing the term number and the breakdown of letters in each term?
- did you look for patterns in that table?
- did you think about how to come up with the numbers in the table, such as the number of Bs, by using the term number?
- did you try to find rules that use the term number to calculate the other numbers in the table?
- did you check your arithmetic?

If your answer does match ours,

- did you use algebraic techniques to find your answer?
- did you show and explain the thinking and work you did?
- is your explanation clear and complete? Would another student understand your solution?
- did you make any mistakes along the way? If so, how did you find and fix them?
- are there any hints that you would give another student?
- have you tried the Extra question?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did—you might answer one or more of the questions above.

## Our Solutions

**Method 1: Make a Table**

I started by making a chart showing the term number, the number of Bs in that term, the number of As in that term, and the total number of letters in that term:

| Term | # of Bs | # of As | total letters |
|---|---|---|---|
| 1 | 2 | 2 | 4 |
| 2 | 3 | 4 | 7 |
| 3 | 4 | 6 | 10 |
| 4 | 5 | 8 | 13 |
| n | n + 1 | 2n | 3n + 1 |

I looked for patterns in the columns and noticed that the number of Bs was always one more than the number of the term since the first term had two Bs and each term adds one more. That means that for the nth term there would be n + 1 Bs.

I also noticed that the number of As was always twice the number of the term since the first term had two As and each term adds two new ones, one at the beginning and one at the end. That means that for the nth term there would be 2*n or 2n As.

To find the total number of letters in the term, I just added the number of Bs and the number of As, so for the nth term there would be (n + 1) + 2n or 3n + 1 letters.

Now I was ready to answer the three questions:

*Question 1*

As explained above, there will be n + 1 Bs in the nth term.

*Question 2*

Taking my expressions for the number of Bs and the total number of letters in the nth term, I can write the ratio of Bs to total letters as (n + 1):(3n + 1) or $\frac{n+1}{n+3}$.

*Question 3*

If exactly 35% of the letters are Bs, that means the ratio of Bs to total letters in the term must be 35/100 or some equivalent fraction. I'll use 35/100 and set the ratio from question 2 equal to it, then solve for n:

$$\frac{n+1}{n+3} = \frac{35}{100}$$

Cross-multiplying the two ratios since they form a proportion, I get an equation I can solve:

$$\begin{aligned} 100(n+1) &= 35(3n+1) \\ 100n + 100 &= 105n + 35 \\ 100 &= 5n + 35 \\ 65 &= 5n \\ 13 &= n \end{aligned}$$

The 13th term will have exactly 35% Bs. I can check this by substituting n = 13 into my two expressions and finding that there will be 13 + 1 or 14 Bs out of 3*13 + 1 or 40 letters. That makes $\frac{14}{40}$ or $\frac{7}{20}$ or $\frac{35}{100}$, which is 35%. Or I can divide 14 by 40 and multiply by 100 to get a percent. $\frac{14}{40}$ is 0.35, and multiplying by 100 makes exactly 35%.

**Extra**

Using a similar approach as in question 3, if the total number of Bs in the term is $\frac{1}{3}$ of the total letters, then the ratio of Bs to total letters is 1:3 and I can set that equal to the ratio from question 2:

$$\frac{n+1}{n+3} = \frac{1}{3}$$

Again I have a proportion, so I can cross-multiply and get:

$$3(n+1) = 1(3n+1)$$
$$3n+3 = 3n+1$$
$$3 = 1$$
$$2 = 0$$

This equation has no solution since 2 does not equal 0. There is no value of n that satisfies the equation, so there is no term in the pattern that will have $\frac{1}{3}$ Bs!

I can see why it's impossible to have 1/3 Bs. Since the ratio of Bs to total letters is $\frac{n+1}{n+3}$, it's close to $\frac{1}{3}$ because of the $\frac{n}{3n}$ parts, but adding 1 to both the numerator and denominator will always move it away from $\frac{1}{3}$ because adding the one in the numerator will have a bigger effect than adding it in the denominator since the 3n is a larger number than the n. It seems like the bigger n gets, the closer the ratio gets to $\frac{1}{3}$, but it can never get there. Here's a table calculating the ratio for various values of n as they get larger and larger:

| n | $\frac{n+1}{n+3}$ | decimal form |
|---|---|---|
| 1 | $\frac{2}{4}$ | 0.5 |
| 10 | $\frac{11}{31}$ | 0.354838710 |
| 100 | $\frac{101}{301}$ | 0.335548173 |
| 1000 | $\frac{1001}{3001}$ | 0.333555482 |
| 10000 | $\frac{10001}{30001}$ | 0.333355555 |

It's getting pretty close to $\frac{1}{3}$ or 0.333..., but it will never get there!

**Method 2: Understand the Problem**

I started by making a chart showing the term number, the number of Bs in that term, the total number of letters in that term, and the ratio of Bs to total letters:

| Term | # of Bs | Total Letters | Ratio of Bs to Total |
|---|---|---|---|
| 1 | 2 | 4 | 2/4 (50%) |
| 2 | 3 | 7 | 3/7 (43%) |
| 3 | 4 | 10 | 4/10 (40%) |
| 4 | 5 | 13 | 5/13 (38%) |

I noticed that every time another B is added, three total letters are added. That makes sense since each new term adds a B and two As, one at each end. I multiplied the number of Bs times 3 to see if it made the total number of letters. It didn't, but I noticed it was always 2 too high. So total letters = (number of Bs * 3) - 2.

*Question 1*

I noticed that the number of Bs is always one more than the term number, which makes sense since the first term has two Bs and one more B is added with each term. So in the nth term, there are (n + 1) Bs.

*Question 2*

Since I figured out that the total letters was 3 times the Bs minus 2, the ratio of Bs to total letters in a given term is (B):(3B - 2), or $\frac{B}{3B-2}$. Since that's true in any term, it's true for the nth term.

*Question 3*

I need my ratio from question 2 to be 35%, so I set it equal to 0.35 and solved for B:

$$\frac{B}{3B-2} = \frac{0.35}{1}$$

Since it was a proportion, I cross-multiplied and set the two products equal:

$$\begin{aligned} B(1) &= (0.35)(3B-2) \\ B &= 1.05B - 0.70 \\ 0.70 + B &= 1.05B \\ 0.70 &= 0.05B \\ 14 &= B \end{aligned}$$

That means there are 14 Bs in the term with 35% Bs. Since I know the number of Bs in a term is one more than the term number, it must be that the 13th term has exactly 35% Bs.

**Extra**

As with question 3 above, I can use the ratio from question 2 to determine which term has $\frac{1}{3}$ Bs. I set the ratio equal to $\frac{1}{3}$, cross-multiplied the proportion, and solved for B:

$$\begin{aligned} \frac{B}{3B-2} &= \frac{1}{3} \\ 3B &= 1(3B-2) \\ 3B &= 3B - 2 \\ 0 &= -2 \end{aligned}$$

This equation has no solution since -2 does not equal 0. There is no value of B that satisfies the equation, so there is no term in the pattern that will have $\frac{1}{3}$ Bs!

I can see why it's impossible to have $\frac{1}{3}$ Bs. Since the ratio of Bs to total letters is $\frac{B}{3B-2}$, it's close to $\frac{1}{3}$ because of the $\frac{B}{3B}$ parts, but subtracting 2 from the denominator makes the fraction a little larger, just as $\frac{1}{2}$ is larger than $\frac{1}{4}$ or $\frac{1}{5}$ is larger than $\frac{1}{7}$. It seems like the bigger B gets, the closer the ratio gets to $\frac{1}{3}$, but it can never get there. Here's a table calculating the ratio for various values of B as they get larger and larger:

| B | $\frac{B}{3B-2}$ | decimal form |
|---|---|---|
| 1 | $\frac{1}{1}$ | 1.0 |
| 10 | $\frac{10}{28}$ | 0.357142857 |
| 100 | $\frac{100}{298}$ | 0.335570470 |
| 1000 | $\frac{1000}{2998}$ | 0.333555704 |
| 10000 | $\frac{10000}{29998}$ | 0.333355557 |

It's getting pretty close to $\frac{1}{3}$ or 0.333..., but it will never get there!

## Standards

If your state has adopted the Common Core State Standards, this alignment may be helpful:

*Algebra: Create equations that describe numbers or relationships*

1. Create equations and inequalities in one variable and use them to solve problems.

*Algebra: Interpret the structure of expressions*

1. Interpret parts of an expression, such as terms, factors, and coefficients.

*Algebra: Mathematical Practices*

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

## Teaching Suggestions

In *To B or Not to B,* students are charged with examining a series, identifying a pattern and representing it algebraically. Students are asked to find the various relationships in the nth term of the series.

Recognizing patterns and being able to generalize them is a key mathematical idea. Starting in the younger grades, we see problems such asking students for the next number in a pattern like 2, 5, 8, 11, ___. Students tend to engage with patterns in different ways – some can see them immediately, while others need to extend them or verbalize them to make sense of them. If students are struggling to make sense of this pattern ask them to try to write out the fifth and sixth terms and then find a partner to compare their terms. If their terms differ, students should try explaining why they think their fifth and sixth term make sense and decide on whose reasoning makes the most sense. Once they've articulated the fifth and sixth terms, they may want to try to now make a chart to represent the As and Bs and start to explore the questions.

## Sample Student Solutions – Focus on Completeness

In the solutions below, I've provided the scores the students would have received in the **Completeness** category of our scoring rubric. For each solution, I've included a comment about why I would score it as shown, as well as what I'd ask the student to work on when they revise their solution to help them move forward with solving the problem or improving their write-up of their work.

| Novice | Apprentice | Practitioner | Expert |
|---|---|---|---|
| Has written very little that tells or shows how they found their answer. | Submitted explanation without work or work without explanation.<br><br>Leaves out enough details that another student couldn't follow or learn from the explanation. | Explains all of the important steps taken to solve the problem.<br><br>Shows equations, formulas, or calculations, if used, used and explains the rationale behind them. | Adds in useful extensions and further explanation of some of the ideas involved<br><br>The additions are helpful, not just "I'll say more to get more credit." |

**Kellie, age 15, Novice**

b/b

Because you're showing the ratio of Bs

*Kellie's written work doesn't give us a whole lot of insight into her thinking, but it is clear she is interested in looking at the ratio of Bs. I would begin by asking her to add the $6^{th}$, $7^{th}$ and $8^{th}$ lines of the pattern and the talk about the number of As and Bs in each and how those are changing.*

**Mike, age 11, Novice**

1. An expression for the number of Bs in the nth term is n+1.

2. An expression for the ratio of Bs to total letters in the nth term is (n+1) : (3n+1) .

| | | | | | | |
|---|---|---|---|---|---|---|
| term | 1 | 2 | 3 | 4 | n |
| amount of B | 2 | 3 | 4 | 5 | n+1 |
| amount of A | 2 | 4 | 6 | 8 | 2n |
| total of letters | 4 | 7 | 10 | 13 | 3n+1 |

Total of letters is amount of B + amount of A . (n+1)+(2n)=3n+1
The ratio of Bs to total letters in the nth term (n+1) : (3n+1)

35%=35/100

*Mike has shown a nice approach to the problem and organized his thinking about the problem well. What I would like to know more about how he new what information needed to be organized and how is organizational tool helped him make sense of the problem. I'd ask him if he started with the chart right away or tried anything else first? I'd also ask him to talk about how was able to move from the numbers to the variables.*

**Terry, age 16, Apprentice**

to the 10th power 10: 34 Line 6

1. 2 to to the 10th power should be the correct number.

2. Keep on adding three to each line, in the ninh line it would be 34 Bs.

3. There are not to many bs in the 6th line.

1,2:4 2,3:7 3,4:10 4,5:13 n,(n plus1):(3n plus 1)

*Terry's work suggests he is able to see some patterns in the problem, but it's hard to follow how he made sense of those patterns. I'd ask him to tell me how he knew to raise 2 to the $10^{th}$ power and what that represents. He also mentions "adding three to each line" and I'd ask him "three what?"*

**Krishan, age 15, Apprentice**

(n plus 1),(n plus 1):(3n plus 1), n=13, no

1,2:4

2,3:7

3,4:10

4,5:13

n,(n plus1):(3n plus 1)

*Krishan has organized his thinking in a patterned way that allows him to come to a solution but is harder for others to follow. I'd ask him if he could describe what each number represents in his list and perhaps if he could label the 'columns.' I'd then ask him to share more about how he was able to move from the 5th row to the variables and if he could show me the patterns he saw that helped.*

**Young, age 12, Practitioner**

1.n+1 2.(n+1)/(3n+1) 3.The 13th term has exactly 35% Bs

I first started with finding the ratios of the four 'sentences'. Then, I found out how much Bs and As there were in the nth term by seeing the letters. 1.

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1.ABBA          no. of Bs=2
2.AABBBAA       no. of Bs=3
3.AAABBBBAAA    no. of Bs=4
4.AAAABBBBBAAAA no. of Bs=5
n.       etc.   no. of Bs=n+1
```

The number of Bs in the nth term is n+1

2. The number of Bs in the nth term is n+1

The number of As in the nth term is 2n

The total letters in the nth term are (n+1)+2n = 3n+1

The ratio of Bs to total letters in the nth term is (n+1)/(3n+1)

3. (n+1)/(3n+1)=0.35=7/20

20n+20=21n+7

n=13

*Young has done a nice job of supporting his table with descriptions of how he decided to use a table and what the table allowed him to see about the numbers of Bs and As. It is easy to follow his thinking and he explains each step well.*

**Bahar, age 15, Practitioner**

The number of B's in the nth term is n+1, the ratio of B's to the total number of letters is n+1/3n+1, and the 13th term has exactly 35% B's.

| | | | | | |
|---|---|---|---|---|---|
| Stages of N | 1 | 2 | 3 | 4 | n |
| Number of A's | 2(1) | 2(2) | 2(3) | 2(4) | 2(n) |
| Number of B's | 2 | 3 | 4 | 5 | n+1 |
| Total number of Letters | 4 | 7 | 10 | 13 | 3n+1 |
| Ratio of B's to the total number of Letters | 2/4 | 3/7 | 4/10 | 5/13 | n+1/3n+1 |

*Bahar has also explained his work and thinking clearly and has supported his table with a great description of his goals for the problem. He does a nice job of explaining how the table helped him see patterns in the As and Bs. His work is easy to follow and he would also be a Practitioner in Clarity. I'd challenge him to tackle the Extra as well.*

1. Looking at the given patterns helps to figure out that the number of Bs are one more than the stage number. Therefore, the number of Bs are n+1 in the nth stage.

2. The number of As are twice the stage number in each stage. For example there are 6 letters in stage

3. Therefore in the nth stage there are 2n+n+1. That equals 3n+1. The ratio of Bs to total number of letters in the nth term is n+1/3n+1.

3. Since I found the ration of Bs to the total number of letters in the previous question, I can set up the following equation: n+1/3n+1 =35/100 I then cross multiplied to get: 105n+35=100n+100 5n=65 n=13

## Scoring Rubric

A **problem-specific rubric** can be found linked from the problem to help in assessing student solutions. We consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work.

We hope these packets are useful in helping you make the most of Algebra Problems of the Week. Please let me know if you have ideas for making them more useful.

*https://www.nctm.org/contact-us/*